CSIR NET/JRF

Mathematical Science

28 Dec. 2023

PART-A

(Mathematical Sciences)

(1.) Train travel time between stations A and B is 39 hours. Every day a pair of trains leave from A to B and B to A at 6 AM. If the service starts on a Monday, on which earliest day will the same train rakes start the journeys again from their original stations?

(a.) Wednesday

(b.) Thursday

(c.) Friday

(d.) Saturday

(2.) When an alarm goes off, policemen X and Y chase thief T, on foot and on a cycle, respectively, along the same straight road. Initially the distance between X and Y was 4 times that between T and X. If X runs twice as fast as T and Y rides twice as fast as X, then

(a.) X and Y will catch up with T at the same time

(b.) X will catch T first

(c.) Y will catch T first

(d.) Y

(3.) In a family of four, the engineer is the son of the chemist and the brighter of the teacher. The chemist is the wife of the lawyer and the mother of the teacher. Which of the following conclusions is necessarily true?

(a.) The teacher is the sister of the engineer

(b.) The teacher is the son of the chemist

(c.) The lawyer is the father of the teacher

(d.) The lawyer is the brother of the teacher

(4.) Which of the integers 10, 11, 12 and 13 can be written as the sum of squares of four integers (allowing repetition)?

(a.) Only 10

(b.) Only 10 and 11

(c.) Only 10, 11 and 12

(d.) All

(5.) Consider two 24-hour clocks A and B. Clock A gets faster by 8 min and clock B gets slower by 12 min every hour. They are synchronized to the correct time at 05:00 hrs. Within the following

24 hours at a certain instant clock A shows 15:12 hrs and clock B shows 12:12 hrs. What is the true time at the instant?

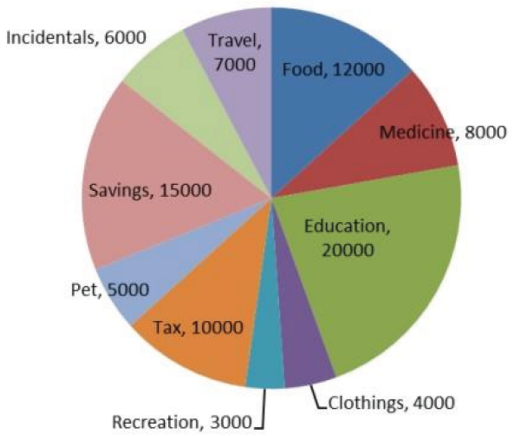
(a.) 13:48

(b.) 14:00

(c.) 14:12

(d.) 14:36

(6.) Average monthly expenses (in rupees) incurred by a family are as shown in the chart.



What is the value of the central angle corresponding to the amount spent on recreation?

(a.) $12°$

(b.) $13°$

(c.) $14°$

(d.) $15°$

(7.) Two cylindrical candles have unequal heights and diameters. The shorter lasts for 13 hours and the longer for 9 hours. They are lit at the same time and after 5 hours their heights are the same. What is the ratio of their original heights?

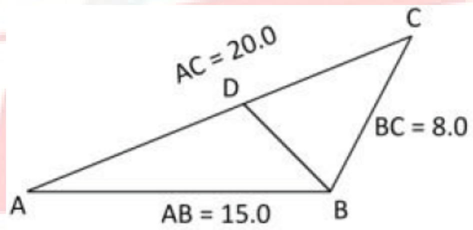
(a.) 1:2

(b.) 13:18

(c.) 9:13

(d.) $\sqrt{5}:3$

(8.) In the figure ΔABC and ΔBDC are similar.



Then $BD = ?$

(a.) 8.0

(b.) 7.2

(c.) 7.5

(d.) 6.2

(9.) From a two-digit number, the sum of its digits is subtracted. The resulting number is

(a.) Always divisible by 6

(b.) Always divisible by 9

(c.) Never divisible by 4

(d.) Never divisible by 5

(10.) Two rectangular pieces of land both having all sides and diagonals in whole numbers in metres have areas in the ratio 4:3 and the smaller (in area) piece has diagonal 41m and one side 9m. However, the bigger piece has a smaller diagonal. The diagonal of the bigger piece is

(a.) 25

(b.) 29

(c.) 32

(d.) 34

(11.) In a queue each women is preceded and followed by exactly two men. A particular women is positioned, from among the women, fourth from the front. The woman's position in the queue from the front is

(a.) 9th

(b.) 10th

(c.) 11th

(d.) 12th

(12.) Which of the following powers of 3 is the largest factor of $1\times2\times3\times4\times\ldots\times30$?

(a.) 3^{10}

(b.) 3^{13}

(c.) 3^{14}

(d.) 3^{15}

(13.) Four students Akash, Bikram, Ramesh and Dewan joined a college in 1991, 1992, 1993 and 1994 but not necessarily in that order. Each student joined one of the four departments, viz, Physics, Chemistry, Mathematics and Biology. No two students joined the same department. One of those who joined the college before 1993 joined Chemistry. No one joined the college after Ramesh. Dewan joined Physics. Akash joined one year after Dewan but didn't join Chemistry. The student who joined in 1992, joined the department of

(a.) Physics

(b.) Chemistry

(c.) Mathematics

(d.) Biology

(14.) Every day a child adds to her piggy bank the same number of coins as are already there in it. If she starts with one coin then the piggy-bank gets full in 8 days. The number of days it will take to fill if she starts with two coins, is

(a.) 4

(b.) 5
(c.) 6
(d.) 7

(15.) In a grid puzzle, each row and column in the 9×9 grid, as well as each 3×3 sub-grid shown with heavy borders, must contain all the digit 1-9.

	8		4		9	6	5	3
6	4	2	8				7	
					?	8		
		7			5		4	2
			7		1			
8	5		6			1		
		6						
	1				4	7	3	6
2	7	3	5	6	8		1	

In the above partially filed grid, the square marked "?" contains
(a.) 2
(b.) 3
(c.) 6
(d.) 7

(16.) Sets $x_1, x_2, \ldots, x_{100}$ and $y_1, y_2, \ldots, y_{150}$ have means zero and the same standard deviations. Which of the following is the ratio of $\sum_1^{100} x_i^2$ to $\sum_1^{150} y_i^2$ closest to?

(a.) 1:1
(b.) $\sqrt{2} : \sqrt{3}$
(c.) 2:3
(d.) 4:9

(17.) Rounding off 4.58500001 to the second decimal place will give
(a.) 4.6
(b.) 4.58
(c.) 4.59
(d.) 4.585

(18.) Consider the equation $3^x - 3^y = 3^4$. A solution to this equation with x and y integers
(a.) Satisfies $x > 4, y > 4$
(b.) Satisfies $x > 5, y > 3$
(c.) Satisfies $x > 6, y > 2$
(d.) Is not possible

(19.) The following 13 observations are molecular weights of 13 compounds (in amu): 65, 61, 63, 65, 61, 60, 65, 83, 65, 84, 61, 65, 62. Which of the following is true of the molecular weights?

(a.) Mean = Median < Mode

(b.) Median < Mode = Mean

(c.) Mode = Median < Mean

(d.) Median < Mean < Mode

(20.) In the following finite sequences of integers, how many terms are divisible by their immediately preceding terms? 8, 3, 4, 9, 3, 5, 9, 5, 9, 9, 9, 4, 5, 6, 3, 3, 5, 7, 2, 3 , 9, 9.

(a.) 3

(b.) 4

(c.) 5

(d.) 6

PART-B

(Mathematical Sciences)

(21.) The value of λ for which the integral equation $y(x)=\lambda\int_0^1 x^2 e^{x+t} y(t)\,dt$ has a non-zero solution, is

(a.) $\dfrac{4}{1+e^2}$

(b.) $\dfrac{2}{1+e^2}$

(c.) $\dfrac{4}{e^2-1}$

(d.) $\dfrac{2}{e^2-1}$

(22.) In any class of 50 students, which one of the following statements is necessarily true?

(a.) Two students have the same birthday

(b.) Every month has birthdays of at least five students

(c.) There exists a month which has birthdays of at least five students

(d.) The birthdays of at least 25 students are during the first six months (from January till June)

(23.) Consider the following subset of $\mathbb{R}$: $U=\{x\in\mathbb{R}: x^2-9x+18\le 0,\ x^2-7x+12\le 0\}$. Which one of the following statements is true?

(a.) $\inf U = 5$

(b.) $\inf U = 5$

(c.) $\inf U = 3$

(d.) $\inf U = 2$

(24.) Which one of the following statements is FALSE?

(a.) The product of two 2×2 real matrices of rank 2 is of rank 2

(b.) The product of two 3×3 real matrices of rank 2 is of rank at most 2 (c.) The product of two 3×3 real matrices of rank 2 is of rank at least 2

(d.) The product of two 2×2 real matrices of rank 1 can be the zero matrix

(25.) Consider the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0,\ -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} e^{\left(-\frac{1}{x^2}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = xe^{-x^2}, x \in \mathbb{R}$$

Which one of the following is true?

(a.) $\lim_{t\to\infty} u(5,t) = 1$

(b.) $\lim_{t\to\infty} u(5,t) = 2$

(c.) $\lim_{t\to\infty} u(5,t) = \frac{1}{2}$

(d.) $\lim_{t\to\infty} u(5,t) = 0$

(26.) Using Euler's method with the step size 0.05, the approximate value of the solution for the initial value problem $\frac{dy}{dx} = \sqrt{3x+2y+1}$, $y(1) = 1$, at $x = 1.1$ (rounded off to two decimal places), is

(a.) 1.50

(b.) 1.65

(c.) 1.25

(d.) 1.15

(27.) The following partial differential equation

$$x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} - 3y^2\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 \text{ is}$$

(a.) Elliptic in $\{(x,y) \in \mathbb{R}^2 : y > 0\}$

(b.) Parabolic in $\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$

(c.) Hyperbolic in $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$

(d.) Parabolic in $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$

(28.) We denote by I_n the $n \times n$ identity matrix. Which one of the following statements is true?

(a.) If A is a real 3×2 matrix and B is a real 2×3 matrix such that $BA = I_2$, then $AB = I_3$

(b.) Let A be the real matrix $\begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$. Then there is a matrix B with integer entries such that $AB = I_2$

(c.) Let A be the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ with entries in $\mathbb{Z}/6\mathbb{Z}$. Then there is a matrix B with entries in $\mathbb{Z}/6\mathbb{Z}$ such that $AB = I_2$

(d.) If A is a real non-zero 3×3 diagonal matrix, then there is a real matrix B such that $AB = I_3$

(29.) In a Latin square design, the degrees of freedom for the sum of squares due to error is 42. Then the degrees of freedom for the sum of squares due to treatments is

(a.) 6

(b.) 7

(c.) 8

(d.) 9

(30.) Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables having the common cumulative distribution function (cdf) $F(x) = \begin{cases} 0, & \text{if } x < 5 \\ 1 - e^{5-x}, & \text{if } x \ge 5 \end{cases}$.

Define $Y_n = \min\{X_1, X_2, \ldots, X_n\}$, $Z_n = \sqrt{n}\,(Y_n - 5)$, $n = 1, 2, \ldots$, and let Z be a standard normal random variable. Then which of the following statements is true?

(a.) $\lim\limits_{n\to\infty} P\left(\frac{1}{2} < Y_n < \frac{3}{2}\right) = 1$

(b.) $Y_n \xrightarrow{P} 5$ as $n \to \infty$

(c.) $Z_n \xrightarrow{d} Z$ as $n \to \infty$

(d.) $\lim\limits_{n\to\infty} P(1 < Z_n < 2) = \Phi(2) - \Phi(1)$, where $\Phi(\cdot)$ denotes the cdf of Z

(31.) Let g denote the acceleration due to gravity and $a > 0$. A particle of mass m glides (without friction) on the cycloid given by $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$, with $0 \le \theta \le 2\pi$. Then the equation of motion of the particle is

(a.) $(1 - \cos\theta)\ddot{\theta} + \frac{1}{2}(\sin\theta)(\dot{\theta})^2 - \frac{g}{2a}\sin\theta = 0$

(b.) $(1 - 2\cos\theta)\ddot{\theta} + (\sin\theta)(\dot{\theta})^2 - \frac{g}{a}\sin\theta = 0$

(c.) $m(1 - 2\cos\theta)\ddot{\theta} + (\sin\theta)(\dot{\theta})^2 + \frac{g}{a}\sin\theta = 0$

(d.) $m(1 - 2\cos\theta)\ddot{\theta} + \frac{m}{2}(\sin\theta)(\dot{\theta})^2 - \frac{g}{a}\sin\theta = 0$

(32.) For $n \ge p + 1$, let $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_n$ be a random sample from $N_p(\underline{\mu}, \Sigma)$, $\underline{\mu} \in \mathbb{R}^p$ and Σ is positive definite matrix. Define $\overline{\underline{X}} = \frac{1}{n}\sum_{i=1}^{n} \underline{X}_i$ and $A = \sum_{i=1}^{n}\left(\underline{X}_i - \overline{\underline{X}}\right)\left(\underline{X}_i - \overline{\underline{X}}\right)^T$. Then the distribution of Trace $(A\Sigma^{-1})$ is (a.) $W_p(n-1,\Sigma)$

(b.) χ_p^2

(c.) χ_{np}^2

(d.) $\chi_{(n-1)p}^2$

(33.) Let $\mathbb{H}=\{z\in\mathbb{C}:\operatorname{Im}(z)>0\}$ denote the upper half plane and let $f:\mathbb{C}\to\mathbb{C}$ be defined by $f(z)=e^{iz}$. Which one of the following statements is true?

(a.) $f(\mathbb{H})=\mathbb{C}\setminus\{0\}$

(b.) $f(\mathbb{H})\cap\mathbb{H}$ is countable

(c.) $f(\mathbb{H})$ is bounded

(d.) $f(\mathbb{H})$ is a convex subset of $\mathbb{C}$

(34.) For $n\geq 2$, let $X_1,X_2,\dots,X_n$ be a random sample from a distribution with the probability density function $f(x\mid\theta)=\begin{cases}\theta x^{\theta-1}, & 0<x<1\\ 0, & \text{otherwise}\end{cases}$, where $\theta>0$ is an unknown parameter. Then which of the following is the uniformly minimum variance unbiased estimator for $\frac{1}{\theta}$?

(a.) $-\frac{1}{n}\sum_{i=1}^{n}\ln X_i$

(b.) $-\frac{n}{\sum_{i=1}^{n}\ln X_i}$

(c.) $-\frac{n-1}{\sum_{i=1}^{n}\ln X_i}$

(d.) $-\frac{2}{n}\sum_{i=1}^{n}\ln X_i$

(35.) Suppose $X\sim\text{Poisson}\left(\frac{3}{4}\right)$. Then which of the following statements is true?

(a.) $P(X>9)\geq\frac{11}{12}$

(b.) $P(X<9)\geq\frac{11}{12}$

(c.) $E\left(X-\frac{3}{4}\right)^2\geq\frac{11}{12}$

(d.) $\frac{11}{9}X\sim\text{Poisson}\left(\frac{11}{12}\right)$

(36.) Let $f:\mathbb{C}\to\mathbb{C}$ be a real-differentiable function. Define $u,v:\mathbb{R}^2\to\mathbb{R}$ by $u(x,y)=\operatorname{Re} f(x+iy)$ and $v(x,y)=\operatorname{Im} f(x+iy)$, $x,y\in\mathbb{R}$. Let $\nabla u=(u_x,u_y)$ denote the gradient. Which one of the following is necessarily true?

(a.) For $c_1,c_2\in\mathbb{C}$, the level curves $u=c_1$ and $v=c_2$ are orthogonal wherever they intersect

(b.) $\nabla u\cdot\nabla v=0$ at every point

(c.) If f is an entire function, then $\nabla u\cdot\nabla v=0$ at every point

(d.) If $\nabla u\cdot\nabla v=0$ at every point, then f is an entire function

(37.) Which one of the following is equal to $1^{37}+2^{37}+3^{37}+\ldots+88^{37}$ in $\mathbb{Z}/89\mathbb{Z}$?

(a.) 88

(b.) -88

(c.) -2

(d.) 0

(38.) The smallest real number λ for which the problem $-y''+3y=\lambda y$, $y(0)=0$, $y(\pi)=0$ has a non-trivial solution is

(a.) 3

(b.) 2

(c.) 1

(d.) 4

(39.) For $n\ge 2$, let $\epsilon_1,\epsilon_2,\ldots,\epsilon_n$ be independent and identically distributed (i.i.d.) $N(0,\sigma^2)$ random variables and $Y_i=i\alpha+i^2\alpha^2+\epsilon_i$, $i=1,\ldots,n$, where $\sigma>0$ and $\alpha\in\mathbb{R}$ are unknown parameters. Then which of the following is a jointly minimal sufficient statistic for (α,σ)?

(a.) $\left(\sum_{i=1}^{n}Y_i^2,\sum_{i=1}^{n}iY_i,\sum_{i=1}^{n}i^2Y_i\right)$

(b.) $\left(\sum_{i=1}^{n}Y_i^2,\sum_{i=1}^{n}iY_i,\sum_{i=1}^{n}i^2Y_i^2\right)$

(c.) $\left(\sum_{i=1}^{n}iY_i,\sum_{i=1}^{n}i^2Y_i^2\right)$

(d.) $\left(\sum_{i=1}^{n}Y_i,\sum_{i=1}^{n}iY_{ia}\right)$

(40.) Let f be a meromorphic function on an open set containing the unit circle C and its interior. Suppose that f has no zeros and no poles on C, and let n_p and n_0 denote the number of poles and zeroes of f inside C, respectively. Which one of the following is true?

(a.) $\frac{1}{2\pi i}\int_C\frac{(zf)'}{zf}dz=n_0-n_p+1$

(b.) $\frac{1}{2\pi i}\int_C \frac{(zf)'}{zf}dz = n_0 - n_p - 1$

(c.) $\frac{1}{2\pi i}\int_C \frac{(zf)'}{zf}dz = n_0 - n_p$

(d.) $\frac{1}{2\pi i}\int_C \frac{(zf)'}{zf}dz = n_p - n_0$

(41.) How many roots does the polynomial $z^{100} - 50z^{30} + 40z^{10} + 6z + 1$ have in the open disc $\{z \in \mathbb{C} : |z| < 1\}$?

(a.) 100
(b.) 50
(c.) 30
(d.) 0

(42.) Consider a linear regression model $Y = \alpha + \beta x + \varepsilon$, where α and β are unknown parameters, and ε is a random error with mean 0. Based on 10 independent observations (x_i, y_i), $i = 1,\ldots,10$, the fitted model, using OLS is $\hat{y}_i = 1.5 + 0.8x_i$, $i = 1,2,\ldots,10$. Suppose that $\sum_{i=1}^{10}\left(y_i - \frac{1}{10}\sum_{j=1}^{10} y_j\right)^2 = 5$ and $\sum_{i=1}^{10}\left(x_i - \frac{1}{10}\sum_{j=1}^{10} x_j\right)^2 = 6$. Then the adjusted coefficient of determination (adjusted R^2) is equal to (after rounding off to two places of decimal)

(a.) 0.74
(b.) 0.83
(c.) 0.77
(d.) 0.84

(43.) Consider the field $\mathbb{C}$ together with the Euclidean topology. Let K be a proper subfield of $\mathbb{C}$ that is not contained in $\mathbb{R}$. Which one of the following statements is necessarily true?

(a.) K is dense in $\mathbb{C}$
(b.) K is an algebraic extension of $\mathbb{Q}$
(c.) $\mathbb{C}$ is an algebraic extension of K
(d.) The smallest closed subset of $\mathbb{C}$ containing K is NOT a field

(44.) Consider the following infinite series:

A. $\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{\sqrt{n}}$

B. $\sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n^2}\right)$

Which one of the following statement is true?

(a.) A is convergent, but B is not convergent
(b.) A is not convergent, but B is convergent
(c.) Both A and B are convergent

(d.) Neither A nor B is convergent

(45.) For $a \in \mathbb{R}$, let $A_a = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & a \end{pmatrix}$. Which one of the following statements is true?

(a.) A_a is positive definite for all $a < 3$

(b.) A_a is positive definite for all $a > 3$

(c.) A_a is positive definite for all $a \geq -2$

(d.) A_a is positive definite only for finitely many values of a

(46.) Let (X,Y) be a random vector with the joint moment generating function

$M_{X,Y}(t_1,t_2) = \left(\frac{3}{4} + \frac{1}{4}e^{t_1}\right)^2 \left(\frac{5}{6} + \frac{1}{6}e^{t_2}\right)^3$, $(t_1,t_2) \in \mathbb{R}^2$. Then $P(X + 2Y > 1)$ is equal to

(a.) $\frac{1581}{3456}$

(b.) $\frac{1875}{3456}$

(c.) $\frac{125}{3456}$

(d.) $\frac{3331}{3456}$

(47.) Let $(-,-)$ be a symmetric bilinear form on $\mathbb{R}^2$ such that there exist non-zero $v, w \in \mathbb{R}^2$ such that $(v,v) > 0 > (w,w)$ and $(v,w) = 0$. Let A be the 2×2 real symmetric matrix representing this bilinear form with respect to the standard basis. Which one of the following statements is true?

(a.) $A^2 = 0$

(b.) Rank $A = 1$

(c.) Rank $A = 0$

(d.) There exists $u \in \mathbb{R}^2$, $u \neq 0$ such that $(u,u) = 0$

(48.) Let $X_1, X_2, \ldots, X_6$ be a random sample from a gamma distribution with the probability density function $f(x \mid \lambda) = \begin{cases} \frac{\lambda^4}{6} e^{-\lambda x} x^3, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$, where $\lambda > 0$ is unknown. Let $T = \sum_{i=1}^{6} X_i$ and ψ be the uniformly most powerful test of size $\alpha = 0.05$ for testing null hypothesis $H_0 : \lambda = 1$ against alternative hypothesis $H_1 : \lambda > 1$. For any positive integer v, let $\chi^2_{v,\alpha}$ denote the $(1-\alpha)^{th}$ quantile of χ^2_v distribution. Then the test ψ rejects H_0 if and only if

(a.) $T \geq \frac{1}{2}\chi^2_{48,0.05}$

(b.) $T \le \frac{1}{2}\chi^2_{48,0.95}$

(c.) $T \ge \frac{1}{2}\chi^2_{24,0.05}$

(d.) $T \le \frac{1}{2}\chi^2_{24,0.95}$

(49.) The probability of getting a head in tossing of a coin is p, $p \in (0,1)$. The coin is independently tossed 25 times and head appears 10 times. The Bayes estimate of p, with respect to the prior $Beta(5,5)$ and the squared error loss function, is

(a.) $\frac{3}{7}$

(b.) $\frac{3}{5}$

(c.) $\frac{1}{2}$

(d.) $\frac{2}{5}$

(50.) Let $A = (a_{i,j})$ be the $n \times n$ real matrix with $a_{i,j} = ij$ for all $1 \le i, j \le n$. If $n \ge 3$, which one of the following is an eigenvalues of A?

(a.) 1

(b.) π

(c.) $n(n+1)/2$

(d.) $n(n+1)(2n+1)/6$

(51.) The cardinality of the set of extremals of $J[y] = \int_0^1 (y')^2\,dx$, subject to $y(0)=1$, $y(1)=6$, $\int_0^1 y\,dx = 3$ is

(a.) 0

(b.) 1

(c.) 2

(d.) Countably infinite

(52.) Let G be any finite group. Which one of the following is necessarily true?

(a.) G is a union of proper subgroups

(b.) G is a union of proper subgroups if $|G|$ has at least two distinct prime divisors

(c.) If G is abelian, then $|G|$ is a union of proper subgroups

(d.) G is a union of proper subgroups if and only if $|G|$ is not cyclic

(53.) Consider the linear programming problem:

Maximize $z = 3x + 4y$

Subject $x + y \le 12$

$2x + 3y \le 30$

$x + 4y \le 36$

$x \ge 0, y \ge 0$

Then the optimal solution of the given problem is

(a.) $x^* = 6, y^* = 6$

(b.) $x^* = 7, y^* = 5$

(c.) $x^* = 3, y^* = 8$

(d.) $x^* = 4, y^* = 8$

(54.) Let A be an $n \times n$ matrix with complex entries. If $n \ge 4$, which one of the following statements is true?

(a.) A does not have any non-zero invariant subspace in $\mathbb{C}^n$

(b.) A has an invariant subspace in $\mathbb{C}^n$ of dimension $n-3$

(c.) All eigenvalues of A are real numbers

(d.) A^2 does not have any invariant subspace in $\mathbb{C}^n$ of dimension $n-1$

(55.) Consider a homogenous Markov chain with state space $\{0,1,2\}$ and transition probability matrix (TPM) given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Let $P^{(n)} = \left(\left(P_{ij}^{(n)}\right)\right)$ by the n-step TPM. Then which of the following statements is true?

(a.) $\lim\limits_{n\to\infty} P_{21}^{(n)} = 1$

(b.) The unique stationary distribution of the chain is given by $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

(c.) $\{1, 2\}$ forms a closed set of states

(d.) $\lim\limits_{n\to\infty} P_{22}^{(n)} = 1$

(56.) Let $f(x)$ be a cubic polynomial with real coefficients. Suppose that $f(x)$ has exactly one real root and that this root is simple. Which one of the following statements holds for ALL anti-derivatives $F(x)$ of $f(x)$

(a.) $F(x)$ has exactly one real root

(b.) $F(x)$ has exactly four real roots

(c.) $F(x)$ has at most two real roots

(d.) $F(x)$ has at most one real root

(57.) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=\begin{cases}(1-x)^2\sin(x^2), & x\in(0,1)\\ 0, & \text{otherwise}\end{cases}$ and f' be its derivative. Let $S=\{c\in\mathbb{R}: f'(x)\le cf(x) \text{ for all } x\in\mathbb{R}\}$. Which one of the following is true?

(a.) $S=\emptyset$

(b.) $S\neq\emptyset$ and S is a proper subset of $(1,\infty)$

(c.) $(2,\infty)$ is a proper subset of S

(d.) $S\cap(0,1)\neq\emptyset$

(58.) Let X be an non-empty finite set and $Y=\{f^{-1}(0): f$ is a real-valued function on $X\}$. Which one of the following statements is true?

(a.) Y is an infinite set

(b.) Y has $2^{|X|}$ elements

(c.) There is bijective function from X to Y

(d.) There is a surjective function from X to Y

(59.) Let $f:\mathbb{R}\to\mathbb{R}$ be an differentiable function such that f and its derivative f' have no common zeros in $[0,1]$. Which one of the following statements is true?

(a.) f never vanishes in $[0,1]$

(b.) f has at most finitely many zeroes in $[0,1]$

(c.) f has infinitely many zeros in $[0,1]$

(d.) $f\left(\frac{1}{2}\right)=0$

(60.) Consider the sequence $(a_n)_{n\ge1}$, where $a_n=\cos\left((-1)^n\frac{n\pi}{2}+\frac{n\pi}{3}\right)$. Which one of the following statements is true?

(a.) $\limsup_{n\to\infty} a_n=\frac{\sqrt{3}}{2}$

(b.) $\limsup_{n\to\infty} a_{2n}=1$

(c.) $\limsup_{n\to\infty} a_{2n}=\frac{1}{2}$

(d.) $\limsup_{n\to\infty} a_{3n}=0$

PART-C

(Mathematical Sciences)

(61.) Let X_i be an absolutely continuous random variable having the probability density function

$$f_i(x)=\begin{cases} ie^{-ix}, & \text{if } x\ge 0 \\ 0, & \text{if } x<0 \end{cases}, i=1,2.$$

Consider a series system comprising of independent components having random lifetimes described by random variables X_1 and X_2. Let X denote the lifetime of the series system. Then which of the following statements are true?

(a.) $P(X>4)=P(X>1)P(X>2)$

(b.) $P(X>4\,|\,X>2)=P(X>2)$

(c.) $E(X)=\dfrac{1}{3}$

(d.) $6X \sim \chi_3^2$

(62.) Consider the one-way fixed effects ANOVA model

$Y_{ij}=\mu+\alpha_i+\varepsilon_{ij}$, $j=1,\ldots,n_i$; $i=1,\ldots,k$,

where the errors $\varepsilon_{ij}s$ are uncorrelated with mean 0 and finite variance $\sigma^2(>0)$. Let $Y_i=\dfrac{1}{n_i}\sum_{j=1}^{n_i}Y_{ij}$ for $i=1,\ldots,k$. Then, which of the following statements are true?

(a.) $\dfrac{1}{\sum_{i=1}^{k}n_i}\sum_{i=1}^{k}\sum_{j=1}^{n_i}Y_{ij}$ is an unbiased estimator of μ

(b.) $2\mu+\alpha_1+\alpha_2$ is an estimable linear parametric function

(c.) $\mu+\alpha_1+\alpha_2$ is an estimable linear parametric function

(d.) $\dfrac{1}{n_2}\sum_{j=1}^{n_2}\left(Y_{2j}-\bar{Y}_2\right)$ is an unbiased estimator of α_2

(63.) Consider the following initial value problem $y'=y+\dfrac{1}{2}\left|\sin\left(y^2\right)\right|$, $x>0$, $y(0)=-1$. Which of the following statements are true?

(a.) There exists an $\alpha\in(0,\infty)$ such that $\lim_{x\to\alpha^-}|y(x)|=\infty$

(b.) $y(x)$ exist on $(0,\infty)$ and it is monotone

(c.) $y(x)$ exist on $(0,\infty)$, but not bounded below

(d.) $y(x)$ exist on $(0,\infty)$, but not bounded above

(64.) Consider the Cauchy problem

$u\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}=1$, $(x,y)\in\mathbb{R}\times(0,\infty)$,

$u(x,0)=kx$, $x\in\mathbb{R}$,

with a given real parameter k. For which of the following values of k does not above problem have a solution defined on $\mathbb{R}\times(0,\infty)$?

(a.) $k = 0$

(b.) $k = -2$

(c.) $k = 4$

(d.) $k = 1$

(65.) Define $S = \{y \in C^1[0,\pi] : y(0) = y(\pi) = 0\}$

$\|f\|_\infty = \lim\limits_{x \in [0,\pi]} |f(x)|$, for all $f \in S$

$B_0(f,\varepsilon) = \{f \in S : \|f\|_\infty < \varepsilon\}$

$B_1(f,\varepsilon) = \{f \in S : \|f\|_\infty + \|f'\|_\infty < \varepsilon\}$.

Consider the functional $J : S \to \mathbb{R}$ given by $J[y] = \int_0^\pi \left(1 - (y')^2\right) y^2 dx$.

Then there exists $\varepsilon > 0$ such that

(a.) $J[y] \le J[0]$, for all $y \in B_0(0,\varepsilon)$

(b.) $J[y] \le J[0]$, for all $y \in B_1(0,\varepsilon)$

(c.) $J[y] \ge J[0]$, for all $y \in B_0(0,\varepsilon)$

(d.) $J[y] \ge J[0]$, for all $y \in B_1(0,\varepsilon)$

(66.) Consider two groups, say G_1 and G_2, comprising of 10 and 30 patients, respectively. Suppose that mean diastolic blood pressures of patients in groups G_1 and G_2 are 80 mm Hg and 100 mm Hg, respectively, and the corresponding variances are 4 mm Hg^2 and 2 mm Hg^2, respectively. Let $\bar{X}, S^2, C$ and R, respectively, denote the mean (in mm Hg), variance (in mm Hg^2), coefficient of variation (in percentage) and range (in mm Hg) of the diastolic blood pressures of the combined group (the two groups combined). Then which of the following statements are true?

(Note: For observations $x_1, x_2, \dots, x_n$, variance is defined by $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$, where $\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$).

(a.) $\bar{X} = 95$

(b.) $S^2 = 77$

(c.) $C > \frac{180}{19}$

(d.) $R > 8$

(67.) Let $R = \mathbb{Z}[x]/(X^2+1)$ and $\psi : \mathbb{Z}[X] \to R$ be the natural quotient map. Which of the following statements are true?

(a.) R is isomorphic to a subgring of $\mathbb{C}$

(b.) For any prime number $p \in \mathbb{Z}$, the ideal generated by $\psi(p)$ is a proper ideal of R

(c.) R has infinitely many prime ideals

(d.) The ideal generated by $\psi(X)$ is a prime ideal in R

(68.) Consider $\mathbb{R}^2$ with the Euclidean topology and consider $\mathbb{Q}^2 \subset \mathbb{R}^2$ with the subspace topology. Which of the following statements are true?

(a.) $\mathbb{Q}^2$ is connected

(b.) If A is a non-empty connected subset of $\mathbb{Q}^2$, then A has exactly one element

(c.) $\mathbb{Q}^2$ is Hausdorff

(d.) $\{(x,y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\}$ is compact in the subspace topology

(69.) Let $f(X) = X^2 + X + 1$ and $g(X) = X^2 + X - 2$ be polynomials in $\mathbb{Z}[X]$. Which of the following statements are true?

(a.) For all prime numbers p, $f(X) \bmod p$ is irreducible in $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)[X]$

(b.) There exists a prime number p such that $g(X) \bmod p$ is irreducible in $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)[X]$

(c.) $g(X)$ is irreducible in $\mathbb{Q}[X]$

(d.) $f(X)$ is irreducible in $\mathbb{Q}[X]$

(70.) Let y be the solution to the Volterra integral equation $y(x) = e^x + \int_0^x \frac{1+x^2}{1+t^2} y(t)\,dt$.

Then which of the following statements are true?

(a.) $y(1) = \left(1 + \frac{\pi}{4}\right)e$

(b.) $y(1) = \left(1 + \frac{\pi}{2}\right)e$

(c.) $y(\sqrt{3}) = \left(1 + \frac{3\pi}{4}\right)e^{\sqrt{3}}$

(d.) $y(\sqrt{3}) = \left(1 + \frac{4\pi}{3}\right)e^{\sqrt{3}}$

(71.) Suppose a 7×7 block diagonal complex matrix A has blocks $(0), (1), \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 2\pi i & 1 & 0 \\ 0 & 2\pi i & 0 \\ 0 & 0 & 2\pi i \end{pmatrix}$ along the diagonal. Which of the following statements are true?

(a.) The characteristic polynomial of A is $x^3(x-1)(x-2\pi i)^3$

(b.) The minimal polynomial of A is $x^2(x-1)(x-2\pi i)^3$

(c.) The dimensions of the eigenspaces for $0, 1, 2\pi i$ are 2,1,3 respectively

(d.) The dimensions of the eigenspaces for $0, 1, 2\pi i$ are 2,1,2 respectively

(72.) The coefficient of x^3 in the interpolating polynomial for the data

x	0	1	2	3	4
y	1	2	1	3	5

is

(a.) $-\frac{1}{3}$

(b.) $-\frac{1}{2}$

(c.) $\frac{5}{6}$

(d.) $\frac{17}{6}$

(73.) Consider the quadratic form $Q(x,y,z) = x^2 + xy + y^2 + xz + yz + z^2$. Which of the following statements are true?

(a.) There exists a non-zero $u \in \mathbb{Q}^3$ such that $Q(u) = 0$

(b.) There exists a non-zero $u \in \mathbb{R}^3$ such that $Q(u) = 0$

(c.) There exists a non-zero $u \in \mathbb{C}^3$ such that $Q(u) = 0$

(d.) The real symmetric 3×3 matrix A which satisfies $Q(x,y,z) = [x\ y\ z]A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for all $x,y,z \in \mathbb{R}$ is invertible

(74.) Let $(f_n)_{n\ge1}$ be the sequence of functions defined on $[0,1]$ by $f_n(x) = x^n \log\left(\frac{1+\sqrt{x}}{2}\right)$. Which of the following statements are true?

(a.) (f_n) converges pointwise on $[0,1]$

(b.) (f_n) converges uniformly on compact subset of $[0,1)$ but not on $[0,1)$

(c.) (f_n) converges uniformly on $[0,1)$ but not on $[0,1]$

(d.) (f_n) converges uniformly on $[0,1]$

(75.) Suppose that $f:[-1,1] \to \mathbb{R}$ is continuous. Which of the following imply that f is identically zero on $[-1,1]$?

(a.) $\int_{-1}^{1} f(x)x^n dx = 0$ for all $n \ge 0$

(b.) $\int_{-1}^{1} f(x)p(x)dx = 0$ for all real polynomials $p(x)$

(c.) $\int_{-1}^{1} f(x)x^n dx = 0$ for all $n \ge 0$ odd

(d.) $\int_{-1}^{1} f(x)x^n dx = 0$ for all $n \ge 0$ even

(76.) Let $\mathbb{F}$ be a finite field and V be a finite dimensional non-zero $\mathbb{F}$-vector space. Which of the following can NEVER be true?

(a.) V is the union of 2 proper subspaces

(b.) V is the union of 3 proper subspaces

(c.) V has a unique basis

(d.) V has precisely two bases

(77.) Consider the initial value problem $x^2y'' - 2x^2y' + (4x-2)y = 0$, $y(0) = 0$. Suppose $y = \varphi(x)$ is a polynomial solution satisfying $\varphi(1) = 1$. Which of the following statements are true?

(a.) $\varphi(4) = 16$

(b.) $\varphi(2) = 2$

(c.) $\varphi(5) = 25$

(d.) $\varphi(3) = 3$

(78.) Among the curves connecting the points $(1, 2)$ and $(2, 8)$, let γ be the curve on which an extremal of the functional $J[y] = \int_1^2 (1 + x^3y')y'dx$ can be attained. Then which of the following points lie on the curve γ?

(a.) $(\sqrt{2}, 3)$

(b.) $(\sqrt{2}, 6)$

(c.) $\left(\sqrt{3}, \frac{22}{3}\right)$

(d.) $\left(\sqrt{3}, \frac{23}{3}\right)$

(79.) For an integer k, consider the contour integral $I_k = \int_{|Z|=1} \frac{e^z}{z^k} dz$. Which of the following statements are true?

(a.) $I_k = 0$ for every integer k

(b.) $I_k \neq 0$ if $k \geq 1$

(c.) $|I_k| \leq |I_{k+1}|$ for every integer k

(d.) $\lim_{k\to\infty} |I_k| = \infty$

(80.) For a differentiable surjective function $f : (0,1) \to (0,1)$, consider the function $F : (0,1)\times(0,1) \to (0,1)\times(0,1)$ given by $F(x, y) = (f(x), f(y))$, $x, y \in (0,1)$. If $f'(x) \neq 0$ for every $x \in (0,1)$, then which of the following statements are true?

(a.) F is injective

(b.) f is increasing

(c.) For every $(x', y') \in (0,1) \times (0,1)$, there exists a unique $(x, y) \in (0,1) \times (0,1)$ such that $F(x, y) = (x', y')$

(d.) The total derivative $DF(x, y)$ is invertible for all $(x, y) \in (0,1) \times (0,1)$

(81.) Consider an M/M/1 queuing model with arrival rate $\lambda = 15$ per hour and service rate $\mu = 45$ per hour. Let $N(t)$ denote the number of customers in the system at time $t \in (0, \infty)$. Also let T_1 and T_2 be the amounts of time a customer spends in the queue and in the system, respectively. Then which of the following statements are true?

(a.) $\lim_{t \to \infty} P(N(t) = 1) = \frac{2}{9}$

(b.) $P(T_1 > 0) = \frac{1}{3}$

(c.) $P(T_1) = \frac{1}{90}$

(d.) $P(T_2) = \frac{1}{35}$

(82.) Suppose X is a continuous random variable with probability density function

$$f(x) = \frac{1}{\pi} \frac{1}{1 + (x+1)^2}, \; -\infty < x < \infty. \text{ Define } Y = \begin{cases} \frac{X}{|X|}, & \text{if } X \neq 0 \\ 0, & \text{if } X = 0 \end{cases}$$

Then which of the following statements are true?

(a.) $E(Y) = 0$

(b.) $P(Y > 0) < P(Y < 0)$

(c.) $P(Y < -1) < P(Y > 1)$

(d.) $E(Y^2) = 1$

(83.) Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is unknown. In order to test $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$, where $\mu_0 \in \mathbb{R}$ is some specific constant, consider the following two tests:

(A) Reject H_0 if and only if $\bar{X}_n > c_1$, where c_1 is such that $P_{\mu_0}(\bar{X}_n > c_1) = \alpha \in (0,1)$ and $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

(B) Reject H_0 if and only if Median $\{X_1, \ldots, X_n\} > c_2$, where c_2 is such that P_{μ_0} (median $\{X_1, \ldots, X_n\} > c_2)$ $= \alpha \in (0,1)$.

Then which of the following statements are true?

(a.) The test described in (A) is the uniformly most powerful test of size α

(b.) The test described in (B) is the uniformly most powerful test of size α

(c.) $P_\mu(\bar{X}_n > c_1) \to 1$ as $n \to \infty$ for all $\mu > \mu_0$

(d.) $P_{\mu_0}\left(\text{Median}\{X_1,\ldots,X_n\} > \mu_0\right) = \frac{1}{2}$

(84.) Let $\Omega_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $\Omega_2 = \mathbb{C}$. Which of the following statements are true?

(a.) There exists a holomorphic surjective map $f : \Omega_1 \to \Omega_2$

(b.) There exists holomorphic surjective map $f : \Omega_2 \to \Omega_1$

(c.) There exists holomorphic injective map $f : \Omega_1 \to \Omega_2$

(d.) There exists holomorphic injective map $f : \Omega_2 \to \Omega_1$

(85.) Consider the problem $y' = \left(1-y^2\right)^{20} \cos y$, $y(0) = 0$.

Let J be the maximal interval of existence and K be the range of the solution of the above problem. Then which of the following statements are true?

(a.) $J = \mathbb{R}$

(b.) $K = (-1,1)$

(c.) $J = (-1,1)$

(d.) $K = [-1,1]$

(86.) Let $(X_1,Y_1),(X_2,Y_2)$ and (X_3,Y_3) be independent and identically distributed (i.i.d.) random vectors following a bivariate normal distribution with mean vector $(0,0)$ and correlation matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, where $|\rho| < 1$. Suppose that $S_\rho = 3E\left(\operatorname{sgn}(X_1 - X_2)(Y_1 - Y_3)\right)$, where

$\operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Then which of the following statements are true?

(a.) If X_1 and Y_1 are independent random variables, then $S_\rho = 0$

(b.) $S_\rho = \frac{6}{\pi}\sin^{-1}\frac{\rho}{2}$

(c.) If $S_\rho = 0$ then X_1 and Y_1 are independent random variables

(d.) If X_1 and Y_1 are independent random variables, then $S_\rho = \frac{1}{2}$

(87.) Suppose $U \sim \text{Uniform}(0,1)$, and $X = \tan\left(\pi\left(U - \frac{1}{2}\right)\right)$. Then which of the following statements are true?

(a.) $E\left(X^4\right) = 3$

(b.) $P\left(X \in \{1,2,5\}\right) = \frac{1}{2}$

(c.) $E\left(e^X\right)$ does not exist

(d.) $P(X \leq 0) = \frac{1}{2}$

(88.) Let X, Y be two $n \times n$ real matrices such that $XY = X^2 + X + I$. Which of the following statements are necessarily true?

(a.) X is invertible

(b.) $X + I$ is invertible

(c.) $XY = YX$

(d.) Y is invertible

(89.) Let X be a discrete random variable with the support $S = \{-1, 0, 1\}$ and $P(X = 0) = \frac{1}{3}$. Then which of the following statements are true?

(a.) $E(X) \leq \frac{2}{3}$

(b.) $E(X^2) = \frac{2}{3}$

(c.) $E(|X|) = \frac{2}{3}$

(d.) $Var(X) > \frac{2}{3}$

(90.) Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be a $\mathbb{R}$-linear transformation. Suppose that $(1, -1, 2, 4, 0)$, $(4, 6, 1, 6, 0)$ and $(5, 5, 3, 9, 0)$ span the null space of T. Which of the following statements are true?

(a.) The rank of T is equal to 2

(b.) Suppose that for every vector $v \in \mathbb{R}^5$, there exists n such that $T^n v = 0$. Then T^2 must be zero

(c.) Suppose that for every vector $v \in \mathbb{R}^5$, there exists n such that $T^n v = 0$. Then T^3 must be zero

(d.) $(-2, -8, 3, 2, 0)$ is contained in the null space of T

(91.) Let $n \in \mathbb{Z}$ be such that n is congruent to 1 mod 7 and n is congruent to 4 mod 15. Which of the following statements are true?

(a.) n is congruent to 1 mod 3

(b.) n is congruent to 1 mod 35

(c.) n is congruent to 1 mod 21

(d.) n is congruent to 1 mod 5

(92.) Suppose that $\{X(t) : t \geq 0\}$ and $\{Y(t) : t \geq 0\}$ are two independent homogenous Poisson processes having the same arrival rate $\lambda = 2$. Let W_n^X and W_n^Y be the waiting times for the n^{th} arrival for the processes $\{X(t) : t \geq 0\}$ and $\{Y(t) : t \geq 0\}$, respectively, $n \in \mathbb{N}$. Then which of the following statements are true?

(a.) $P(W_2^X < W_3^Y) = \frac{11}{16}$

(b.) $P\left(W_1^X < W_1^Y\right) = \frac{1}{2}$

(c.) $P\left(W_2^X < W_2^Y\right) = \frac{13}{16}$

(d.) $P\left(W_1^X < W_1^Y\right) = \frac{1}{4}$

(93.) Let X be an uncountable subset of $\mathbb{C}$ and let $f:\mathbb{C}\to\mathbb{C}$ be an entire function. Assume that for every $z\in X$, there exists an integer $n\ge 1$ such that $f^{(n)}(z)=0$. Which of the following statements are necessarily true?

(a.) $f=0$

(b.) f is a constant function

(c.) There exists a compact subset K of $\mathbb{C}$ such that $f^{-1}(K)$ is not compact

(d.) f is a polynomial

(94.) Consider the initial value problem $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$, where f is a twice continuously differentiable function on a rectangle containing the point (x_0,y_0). With the step-size h, let the first iterate of a second order scheme to approximate the solution of the above initial value problem be given by $y_1=y_0+Pk_1+Qk_2$, where $k_1=hf(x_0,y_0)$, $k_2=hf(x_0+\alpha_0 y_0, y_0+\beta_0 k_1)$ and $P,Q,\alpha_0\beta_0\in\mathbb{R}$. Which of the following statements are correct?

(a.) If $\alpha_0=2$, then $\beta_0=2, P=\frac{3}{4}, Q=\frac{1}{4}$

(b.) If $\beta_0=3$, then $\alpha_0=3, P=\frac{5}{6}, Q=\frac{1}{6}$

(c.) If $\alpha_0=2$, then $\beta_0=2, P=\frac{1}{4}, Q=\frac{3}{4}$

(d.) If $\beta_0=3$, then $\alpha_0=3, P=\frac{1}{6}, Q=\frac{5}{6}$

(95.) Let $X_1,X_2,\dots,X_n$ be a random sample from an absolutely continuous distribution with the probability density function $f(x\mid\theta)=\begin{cases} e^{\theta-x}, & \text{if } x\ge\theta \\ 0, & \text{if } x<\theta \end{cases}$, where $\theta\in\mathbb{R}$ is unknown. Define $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ and $X_{(1)}=\min\{X_1,\dots,X_n\}$. Then which of the following statements are true?

(a.) $\bar{X}$ is the method of moments estimator of θ

(b.) $X_{(1)}$ is the maximum likelihood estimator of θ

(c.) $X_{(1)}-\frac{1}{n}$ is the uniformly minimum variance unbiased estimator of θ

(d.) $X_{(1)}$ is a sufficient statistic for θ

(96.) Let $f(X) = X^3 - 2 \in \mathbb{Q}[X]$ and let $K \subset \mathbb{C}$ be the splitting field of $f(X)$ over $\mathbb{Q}$. Let $\omega = e^{2\pi i/3}$. Which of the following statements are true?

(a.) The Galois group of K over $\mathbb{Q}$ is the symmetric group S_3

(b.) The Galois group of K over $\mathbb{Q}(\omega)$ is the symmetric group S_3

(c.) The Galois group of K over $\mathbb{Q}$ is $\mathbb{Z}/3\mathbb{Z}$

(d.) The Galois group of K over $\mathbb{Q}(\omega)$ is $\mathbb{Z}/3\mathbb{Z}$

(97.) Let $f:\mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \ge \log(1 + |x - y|)$ for all $x, y \in \mathbb{R}$. Which of the following statements are true?

(a.) f is necessarily one-one

(b.) f need not be one-one

(c.) f is necessarily onto

(d.) f need not be onto

(98.) Let $p:\mathbb{R}^2 \to \mathbb{R}$ be the function defined by $p(x,y) = x$. Which of the following statements are true?

(a.) Let $A_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Then for each $\gamma \in p(A_1)$, there exists a positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_1)$

(b.) Let $A_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$. Then for each $\gamma \in p(A_2)$, there exists a positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_2)$

(c.) Let $A_3 = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$. Then for each $\gamma \in p(A_3)$, there exists a positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_3)$

(d.) Let $A_4 = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$. Then for each $\gamma \in p(A_4)$, there exists a positive real number ε such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subseteq p(A_4)$

(99.) Consider the multiple linear regression model $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$, where $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)^T$, $\underset{\sim}{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)^T$, $\underset{\sim}{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)^T$, X is a fixed $n \times (p+1)$ matrix $(n > p+1)$ of rank $(p+1)$, and $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed (i.i.d.) $N(0, \sigma^2), (\sigma > 0)$ variables. If $\hat{\underset{\sim}{\beta}}$ is the OLS estimator of $\underset{\sim}{\beta}$, then which of the following statements are true?

(a.) $\frac{1}{\sigma^2}\underset{\sim}{Y}^T X \hat{\underset{\sim}{\beta}}$ has a central χ^2_{p+1} distribution

(b.) $\frac{1}{\sigma^2}\left(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}\right)^T \left(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}\right)$ has a central χ^2_{n-p-1} distribution

(c.) $X\hat{\underset{\sim}{\beta}}$ and $\left(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}\right)^T \left(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}\right)$ are independently distributed

(d.) $\frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ has a central χ^2_{n-1} distribution, where $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

(100.) Consider the following Fredholm integral equation $y(x)-3\int_0^1 txy(t)\,dt = f(x)$, where $f(x)$ is a continuous function defined on the interval $[0,1]$. Which of the following choices for $f(x)$ have the property that the above integral equation admits at least one solution?

(a.) $f(x)=x^2-\dfrac{1}{2}$

(b.) $f(x)=e^x$

(c.) $f(x)=2-3x$

(d.) $f(x)=x-1$

(101.) Consider a population of 3 units having values 2, 4 and 6. A simple random sample (without replacement) of 2 units is to be drawn from the population. Let M denote the sample mean of this sample. Then which of the following statements are true?

(a.) $E(M)=4$

(b.) $E(M^2)=17$

(c.) $E(M^3)=72$

(d.) $Var(M)=1$

(102.) Let $X_1, X_2, \ldots, X_{25}$ be independent and identically distributed (i.i.d.) Bernoulli (p) random variables, with $0<p<1$. Let $\bar{X}=\dfrac{1}{25}\sum_{i=1}^{25} X_i$, $T_1=\begin{cases}\dfrac{5(\bar{X}-0.5)}{\sqrt{\bar{X}(1-\bar{X})}}, & \text{if } 0<\bar{X}<1\\ -5, & \text{if } \bar{X}=0\\ 5, & \text{if } \bar{X}=1\end{cases}$ and $T_2=10(\bar{X}-0.5)$.

For testing $H_0: p=0.5$ against $H_1: p>0.5$, consider two tests ψ_1 and ψ_2 such that ψ_i rejects H_0 if and only if $T_i>2$, $i=1$ and 2. If observed $\bar{X}\in(0.5, 0.75)$, then which of the following statements are true?

(a.) If ψ_1 reject H_0, then ψ_2 also rejects H_0

(b.) If ψ_1 does not reject H_0, then ψ_2 also does not reject H_0

(c.) If ψ_2 rejects H_0, then ψ_1 also rejects H_0

(d.) If ψ_2 does not reject H_0, then ψ_1 also does not reject H_0

(103.) Let A be a real diagonal matrix with characteristic polynomial $\lambda^3-2\lambda^2-\lambda^3+2$. Define a bilinear form $\langle v,w\rangle = v^tAw$ on $\mathbb{R}^3$. Which of the following statements are true?

(a.) A is positive definite

(b.) A^2 is positive definite

(c.) There exists a non-zero $v\in\mathbb{R}^3$ such that $\langle v,v\rangle=0$

(d.) Rank $A=2$

(104.) Let $X_1, X_2, \ldots, X_n$ be a random sample from an unknown distribution with absolutely continuous cumulative distribution function (cdf) F. Let F_0 be a specified absolutely continuous cdf. For testing $H_0 : F(x) = F_0(x)$ for all x against $H_1 : F(x) \neq F_0(x)$ for some x, consider the following two tests statistics: $T_{1,n} = \sup_{x\in\mathbb{R}} \left| \frac{1}{n} \sum_{i=1}^{n} I_{\{X_i \le x\}} - F_0(x) \right|$, and $T_{2,n} = \sup_{x\in\mathbb{R}} n \left| \frac{1}{n} \sum_{i=1}^{n} I_{\{X_i \le x\}} - F_0(x) \right|$, where $I_{\{X_i \le x\}} = \begin{cases} 1, & \text{if } X_i \le x \\ 0, & \text{if } X_i > x \end{cases}$ for $i = 1, 2, \ldots, n$. Then which of the following statements are true?

(a.) $T_{1,n} \xrightarrow{P} 0$ as $n \to \infty$ under H_0

(b.) $T_{2,n} \xrightarrow{P} 0$ as $n \to \infty$ under H_0

(c.) $\lim_{n\to\infty} P_F\left(T_{2,n} > 1\right) = 1$ for all F

(d.) $T_{2,n}$ converges in distribution to a degenerate real valued random variable under H_0

(105.) For a real number λ, consider the improper integrals $I_\lambda = \int_0^1 \frac{dx}{(1-x)^\lambda}$, $K_\lambda = \int_1^\infty \frac{dx}{x^\lambda}$. Which of the following statements are true?

(a.) There exists λ such that I_λ converges, but K_λ does not converge

(b.) There exists λ such that K_λ converges, but I_λ does not converge

(c.) There exists λ such that I_λ, K_λ both converge

(d.) There exists λ such that neither I_λ nor K_λ converges

(106.) Let $f : [0, \infty) \to \mathbb{R}$ be the periodic function of period 1 given by $f(x) = 1 - |2x - 1|$ for $x \in [0, 1]$. Further, define $g : [0, \infty) \to \mathbb{R}$ by $g(x) = f(x)^2$. Which of the following statements are true?

(a.) f is continuous on $[0, \infty)$

(b.) f is uniformly continuous on $[0, \infty)$

(c.) g is continuous on $[0, \infty)$

(d.) g is uniformly continuous on $[0, \infty)$

(107.) Let A be an $n \times n$ real symmetric matrix. Which of the following statements are necessarily true?

(a.) A is diagonalizable

(b.) If $A^k = I$ for some positive integer k, then $A^2 = I$

(c.) If $A^k = 0$ for some positive integer k, then $A^2 = 0$

(d.) All eigenvalues of A are real

(108.) Suppose $A = ((a_{ij})) \sim W_3(5, \Sigma)$, where $\sum = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Then which of the following statements are true?

(a.) $a_{22} \sim X_3^2$

(b.) $\frac{1}{2}a_{22} \sim X_5^2$

(c.) $\frac{1}{33}(a_{11} - 4a_{13} + 4a_{33}) \sim X_3^2$

(d.) $\frac{1}{9}(a_{11} - 4a_{13} + 4a_{33}) \sim X_5^2$

(109.) Which of the following statements are true?

(a.) Let G_1 and G_2 be finite groups such that their orders $|G_1|$ and $|G_2|$ are coprime. Then any homomorphism from G_1 to G_2 is trivial

(b.) Let G be a finite group. Let $f: G \to G$ be a group homomorphism such that f fixes more than half of the elements of G. Then $f(x) = x$ for all $x \in G$

(c.) Let G be a finite group having exactly 3 subgroups. Then G is of order p^2 for some prime p

(d.) Any finite abelian group G has at least $d(|G|)$ subgroups in G, where $d(m)$ denotes the number of positive divisors of m

(110.) Consider $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Suppose $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = aA + bI$ for some $a, b \in \mathbb{Z}$. Which of the following statements are true?

(a.) $a + b > 8$

(b.) $a + b < 8$

(c.) $a + b$ is divisible by 2

(d.) $a > b$

(111.) Let $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be the open unit disc in $\mathbb{R}^2$, $\partial B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be its boundary and $\bar{B} = B \cup \partial B$. For $\lambda \in (0, \infty)$, let S_λ be the set of twice continuously differentiable functions in B, that are continuous on $\bar{B}$ and satisfy

$$\left(\frac{\partial u}{\partial x}\right)^2 + \lambda\left(\frac{\partial u}{\partial y}\right)^2 = 1 \text{ in } B$$

$$u(x,) = 0 \text{ on } \partial B.$$

Then which of the following statements are true?

(a.) $S_1 = \emptyset$

(b.) $S_2 = \emptyset$

(c.) S_1 has exactly one elements and S_2 has exactly two elements

(d.) S_1 and S_2 are both infinite

(112.) Let X be a discrete random variable with support $S_X = \{0,1,2,\ldots,25\}$, and $P(X=x)=\binom{25}{x}\frac{1}{2^{25}}$ for all $x \in S_X$. Then which of the following statements are true?

(a.) The distribution of $X-12.5$ and $12.5-X$ are identical

(b.) $P(X \le 4)=P(X \ge 22)$

(c.) Coefficient of variation (in percentage) of X is 20

(d.) $P(X \le 4.9)=P(X \ge 20.1)$

(113.) Let G be the group (under matrix multiplication) of 2×2 invertible matrices with entries from $\mathbb{Z}/9\mathbb{Z}$. Let a be the order of G. Which of the following statements are true?

(a.) a is divisible by 3^4

(b.) a is divisible by 2^4

(c.) a is not divisible by 48

(d.) a is divisible by 3^6

(114.) Which of the following statements are true?

(a.) The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\begin{cases}[x]\sin\frac{1}{x} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$ has a discontinuity at 0 which is removable

(b.) The function $f:[0,\infty)\to\mathbb{R}$ defined by $f(x)=\begin{cases}\sin(\log x) & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$ has a discontinuity at 0 which is NOT removable

(c.) The function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\begin{cases}e^{1/x} & \text{for } x<0\\ e^{1/(x+1)} & \text{for } x\geq 0\end{cases}$ has a jump discontinuity at 0

(d.) Let $f,g:[0,1]\to\mathbb{R}$ be two functions of bounded variation. Then the product fg has at most countably many discontinuities

(115.) For every $n\geq 1$, consider the entire function $p_n(z)=\sum_{k=0}^{n}\frac{z^k}{k!}$. Which of the following statements are true?

(a.) The sequence of function $(p_n)_{n\geq 1}$ converges to an entire function uniformly on compact subsets of $\mathbb{C}$

(b.) For all $n\geq 1$, p_n has a zero in the set $\{z\in\mathbb{C}:|z|\leq 2023\}$

(c.) There exists a sequence (z_n) of complex numbers such that $\lim_{n\to\infty}|z_n|=\infty$ and $p_n(z_n)=0$ for all $n\geq 1$

(d.) Let S_n denote the set of all the zeroes of p_n. If $a_n=\min_{z\in S_n}|z|$, then $a_n\to\infty$ as $n\to\infty$

(116.) Let x be a real number. Which of the following statements are true?

(a.) There exists an integer $n \geq 1$ such that $n^2 \sin\frac{1}{n} \geq x$

(b.) There exists an integer $n \geq 1$ such that $n\cos\frac{1}{n} \geq x$

(c.) There exists an integer $n \geq 1$ such that $ne^{-n} \geq x$

(d.) There exists an integer $n \geq 2$ such that $n(\log n)^{-1} \geq x$

(117.) For real numbers a,b,c,d,e,f, consider the function $F:\mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x,y)$ $=(ax+by+c,\, dx+ey+f)$, for $x,y \in \mathbb{R}$. Which of the following statements are true?

(a.) F is continuous

(b.) F is uniformly continuous

(c.) F is differentiable

(d.) F has partial derivatives of all orders

(118.) For $n \geq 2$, let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu,\sigma^2)$ population, where $\mu \in (-\infty,\infty)$ and $\sigma > 0$ are unknown. Define $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. For any $\alpha \in (0,1)$ and any positive integer m, let z_α denote the $(1-\alpha)^{th}$ quantile of the standard normal distribution and $t_{m,\alpha}$ denote the $(1-\alpha)^{th}$ quantitle of t-distribution with m degrees of freedom. Then which of the following represent 90% confidence intervals for μ?

(a.) $\left(\bar{X} - \frac{s}{\sqrt{n}} t_{n-1,0.05}, \bar{X} + \frac{s}{\sqrt{n}} t_{n-1,0.05}\right)$

(b.) $\left(\bar{X} - \frac{\sigma}{\sqrt{n}} t_{0.05}, \bar{X} + \frac{\sigma}{\sqrt{n}} t_{0.05}\right)$

(c.) $\left[\bar{X} - \frac{s}{\sqrt{n}} t_{n-1,0.9}, \infty\right)$

(d.) $\left(-\infty,, \bar{X} - \frac{s}{\sqrt{n}} t_{n-1,0.9}\right)$

(119.) Let $\{A_n\}_{n\geq1}$ be a collection of non-empty subsets of $\mathbb{Z}$ such that $A_n \cap A_m = \emptyset$ for $m \neq n$. If $\mathbb{Z} = \cup_{n\geq1} A_n$ then which of the following statements are necessarily true?

(a.) A_n is finite for every integer $n \geq 1$

(b.) A_n is finite for some integer $n \geq 1$

(c.) A_n is infinite for some integer $n \geq 1$

(d.) A_n is countable (finite or infinite) for every integer $n \geq 1$

(120.) Let q_1, q_2 be the generalized coordinates and p_1, p_2 be the conjugate momenta, respectively. Let a and b be such that

$Q_1 = q_1,\ P_1 = ap_1 + 16p_2$

$Q_2 = p_2, \; P_2 = 2q_1 + bq_2$

is a canonical transformation. Then which of the following statements are true?

(a.) $a^2 + b^2 = 2$

(b.) $a - b = 2$

(c.) $a + b = 2$

(d.) $a = 1, b = 1$